Example $(a-2b)^3 = 1\cdot a^3 + 3(a)^2(-2b)^1 + 3(a)^1(-2b)^2 + (-2b)^3$

1

1 1

1 2 1

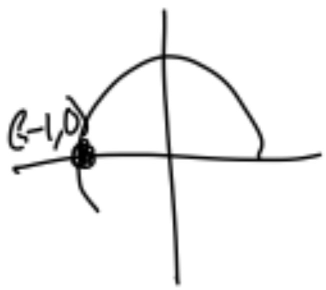
1 3 3 1

$= a^3 - 6a^2b + 12ab^2 - 8b^3$

Example 2.3 (6)

$$\lim_{\theta\to\pi} \sin(2\theta)\cot(\theta)$$

$$= \lim_{\theta\to\pi} \frac{\boxed{\sin(2\theta)}\cos(\theta)}{\sin(\theta)}$$



$\frac{0}{0}$ form.

$$= \lim_{\theta\to\pi} \frac{\boxed{2\cancel{\sin\theta}\underbrace{\cos\theta}_{\to -1}}\underbrace{\cos\theta}_{\to -1}}{(\cancel{\sin\theta})} = \boxed{2}$$

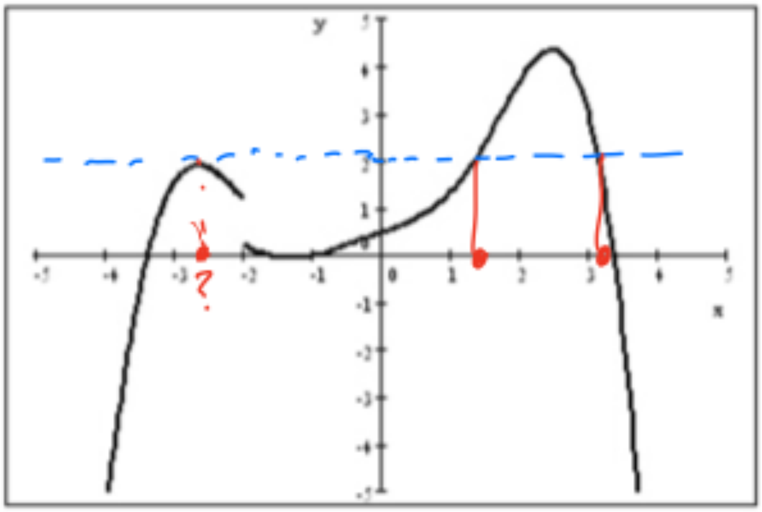
Example

$$C(x) = \begin{cases} 3x^2, & x<1 \\ 4x-1, & x\geq 1 \end{cases}$$

Continuous at $x=1$?

graph

2.4 Consider the graph of $y = g(x)$ below.



(a) Estimate $\lim_{x \to -3^-} g(x)$.

(b) Estimate $\lim_{x \to -2^-} g(x)$.

(c) Estimate $g(3)$.

(d) Find all solutions to the equation $g(x) = 2$.

where is 2 on y graph. $y = g(x)$

What are the x-values when $y = 2$?

Important facts about limits.

① Algebraic Limit theorem.
If f and g are two functions define near $x = c$, and if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} g(x)$ exists,

Then (a) $\lim_{x \to c} (f(x) \pm g(x)) =$

$$\lim_{x \to c} f(x) \pm \lim_{x \to c} g(x)$$

(b) $\lim_{x \to c} (f(x)g(x)) =$

$$\lim_{x \to c} f(x) \lim_{x \to c} g(x)$$

(c) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}$ if

$\lim_{x \to c} g(x) \neq 0.$

(d) $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)}$

etc.

see notes

Proposition 3. *(Algebraic properties of limits) Suppose that f and g are two functions defined near the real number a (or $a = \pm\infty$), and suppose that*

$$\lim_{x\to a} f(x)$$

exists. Then

(1) $\lim\limits_{x\to a} f(x)\,g(x) = \lim\limits_{x\to a} f(x) \lim\limits_{x\to a} g(x)$
(2) $\lim\limits_{x\to a} (f(x) + g(x)) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
(3) $\lim\limits_{x\to a} (f(x) - g(x)) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
(4) $\lim\limits_{x\to a} cf(x) = c \lim\limits_{x\to a} f(x)$ *for any constant real number c*
(5) $\lim\limits_{x\to a} \frac{g(x)}{f(x)} = \frac{\lim_{x\to a} g(x)}{\lim_{x\to a} f(x)}$ *as long as* $\lim\limits_{x\to a} f(x) \neq 0$.
(6) $\lim\limits_{x\to a} c^{f(x)} = c^{\lim_{x\to a} f(x)}$ *for any constant nonzero real number c*
(7) $\lim\limits_{x\to a} h(f(x)) = h\left(\lim\limits_{x\to a} f(x)\right)$ *for any function h continuous at the point* $L = \lim\limits_{x\to a} f(x)$

---

② Order Limit Theorem.

Let f & g be two functions defined near $x = a$. Suppose that $f(x) \le g(x)$ $\forall x$ near $x = a$.

$\forall$ = "for all"

Suppose that $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. Then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

"order is preserved."

Note: The statement is false if $\leq$ is replaced by $<$.

Example $f(x) = 1$, $g(x) = 1 + x^2$.

Note $g(x) > f(x)$ for x near 0.

But $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} 1+x^2 = 1$

$= \lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} 1 = 1$

$g(x) > f(x)$ for x near $x=0$

but $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} f(x)$.

Special Limits

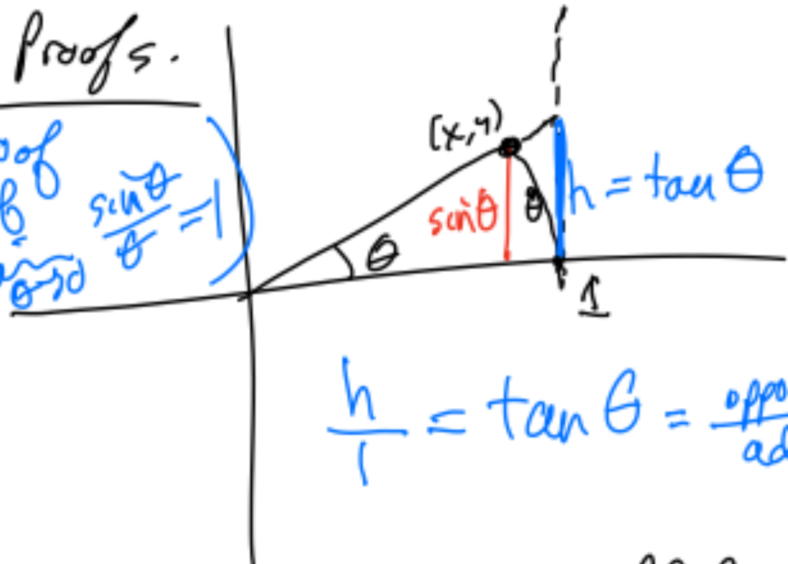
1. $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$

2. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x} = 1$

(2B) $\lim_{y\to\infty}\left(1+\frac{1}{y}\right)^y = e$

Idea of Proofs.

(1) (Proof of $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$)



$\frac{h}{1} = \tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \text{slope}$

from the picture: for small $\theta > 0$

$$0 < \sin\theta \le \theta \le \tan\theta$$

Take reciprocal

$$\frac{1}{\sin\theta} \ge \frac{1}{\theta} \ge \cot\theta = \frac{\cos\theta}{\sin\theta}$$

multiply by $\sin\theta > 0$

$$1 \ge \frac{\sin\theta}{\theta} \ge \cos\theta$$

$$\lim_{\theta\to 0^+} 1 = 1, \quad \lim_{\theta\to 0^+}\cos\theta = 1$$

By the squeeze theorem,

$$\lim_{\theta\to0^+}\frac{\sin\theta}{\theta}=1.$$

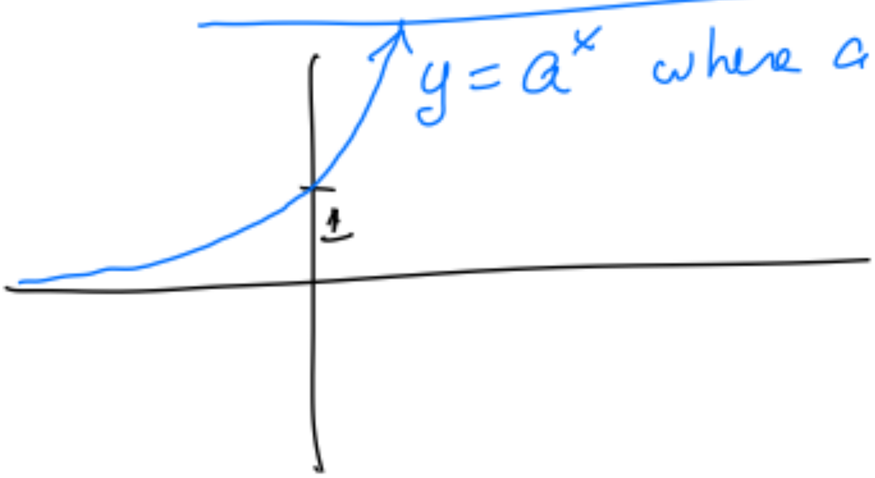
What if $\theta<0$?

$$\lim_{\theta\to0^-}\frac{\sin\theta}{\theta}=\lim_{\theta\to0^-}\frac{-\sin\theta}{-\theta}$$

$$=\lim_{\theta\to0^-}\frac{\sin(\underbrace{-\theta}_{\to0^+})}{\underbrace{-\theta}_{\to0^+}}=1$$ also by previous calc.

$$\therefore\ \boxed{\lim_{\theta\to0}\frac{\sin\theta}{\theta}=1}.$$

② "Proof" of $\lim_{x\to0}\frac{e^x-1}{x}=1.$

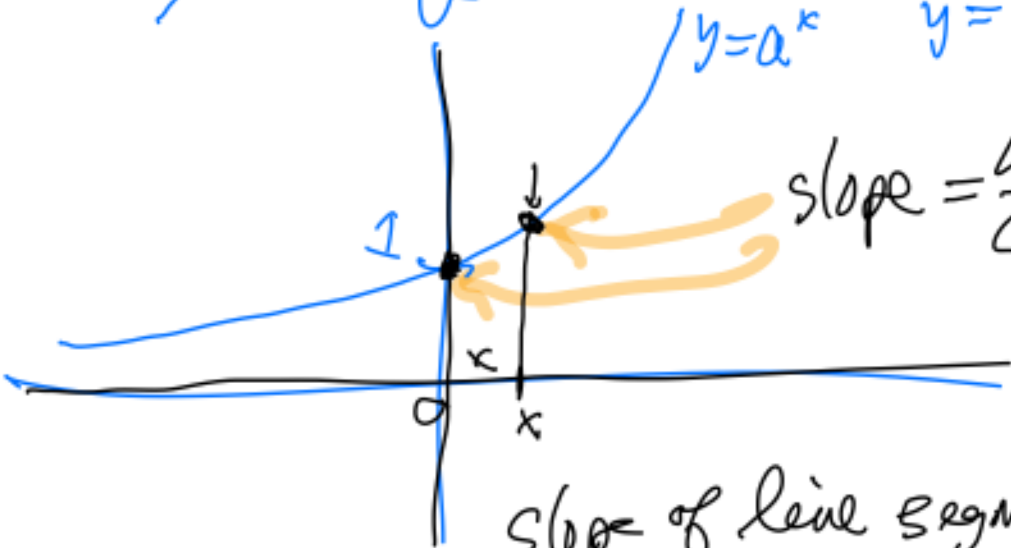


$y=a^x$ where a is a fixed positive #

$a=2$

$a=3.$

Let's compute the slope of the tangent line to $y=a^x$ at $x=0$
$y=1$



$$\text{slope} = \frac{\Delta y}{\Delta x} = \frac{a^x - 1}{x}$$

slope of line segment

$$= \frac{a^x - 1}{x}.$$

$$\left(\begin{array}{c}\text{slope of tangent line} \\ \text{at } x=0 \\ y=1\end{array}\right) = \lim_{x\to 0} \frac{a^x - 1}{x}.$$

From calculation, when $a=2$,

we get tangent slope ≈ 0.69

$a=3$ slope ≈ 1.1

$a=2.5$ slope ≈ 1.03

$a=2.718$ slope ≈ 1.0008

Definition: The number $e \approx 2.718$ is defined to be the number a such that

$$\lim_{x \to 0} \frac{a^x - 1}{x} = 1.$$

$\therefore$ Our special limit (2) $\lim_{x \to 0} \frac{e^x - 1}{x}$ is really the definition of e.

(2B) Proof of $\lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^y = e.$

[Note: this could also be used as the defn of e.]

From (2)

$$\frac{e^x - 1}{x} \approx 1 \quad \text{for very small } x.$$

$$\iff \frac{e^{\frac{1}{y}} - 1}{1/y} \approx 1 \quad \text{for very large } y$$

$\frac{1}{y} \to 0$

$1/y \to 0$

$\iff e^{\frac{1}{y}} - 1 \approx \frac{1}{y}$

multiply by $\frac{1}{y}$

$\iff e^{\frac{1}{y}} \simeq 1 + \frac{1}{y}$

raise to y^{th} power

$\iff \left(e^{\frac{1}{y}}\right)^{y} \simeq \left(1 + \frac{1}{y}\right)^{y}$

$\iff e^{\frac{1}{y} \cdot y} \approx \left(1 + \frac{1}{y}\right)^{y}$

$$e \simeq \left(1 + \frac{1}{y}\right)^{y}.$$

$$\therefore \lim_{y \to \infty} \left(1 + \frac{1}{y}\right)^{y} = e.$$

Examples

① $\lim_{x\to 0} \frac{e^{2x}-1}{x} = ?$

$$\lim_{x\to 0} \frac{e^{2x}-1}{x} = \lim_{x\to 0} \frac{(e^{x}-1)(e^{x}+1)}{x}$$

$\frac{e^x-1}{x} \to 1$, $e^x \to 1$, $e^x+1 \to 2$

$= \boxed{2}.$

or

$$= \lim_{x\to 0} \frac{(e^{2x}-1)}{x} \cdot \frac{2}{2}$$

$$= \lim_{x\to 0} \frac{e^{2x}-1}{2x} \cdot 2 = 1 \cdot 2 = \boxed{2}.$$

$2x \to 0$

$2x \to 0$, $2x \to 0$

$\frac{e^{2x}-1}{2x} \to 1$

special lim

2) $\lim_{x \to \infty} x \sin\left(\frac{3}{x}\right)$

$\frac{\sin \theta}{\theta} \to 1$

$\frac{3}{x} \leftarrow \theta$

$= \lim_{x \to \infty} x \frac{\sin\left(\frac{3}{x}\right)}{\left(\frac{3}{x}\right)}\left(\frac{3}{x}\right)$

$\frac{3}{x} \to 0$, $\frac{3}{x} \to 0$

$3/x \to 0$

$= \lim_{x \to \infty} x \cdot \frac{3}{x} \cdot \frac{\sin\left(\frac{3}{x}\right)}{\left(\frac{3}{x}\right)}$

$= \lim_{x \to \infty} 3 \cdot \frac{\sin\left(\frac{3}{x}\right)}{\left(\frac{3}{x}\right)} = 3 \cdot 1 = \boxed{3}$

$\frac{\sin\left(\frac{3}{x}\right)}{\left(\frac{3}{x}\right)} \to 1$

B)

$\lim_{Bubba \to 0} \frac{\sin(Bubba)}{Bubba} = 1$